Announcements

1) Job talk tomorrow

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3-4 CB 2090

On population dynamics

Recall: Projections onto rank one's, unitaries

If P is an orthogonal projection, $I_n - 2P$ (or $2P - I_n$) is a unitary. Moreover, if $v \in \mathbb{C}^n$ is a unit vector, $P = v \cdot v^*$ is the (rank one) projection onto span(v).

Example 1:

$$A = \begin{bmatrix} 4 & -1 & 0 \\ 3 & 2 & 1 \\ 0 & 5 & 6 \end{bmatrix}$$

Triangularize A by applying unitary matrices (on the left)

This finds R in the QR decomposition.

Do this column-by-column.

Step 1: Send the column $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ to the column $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

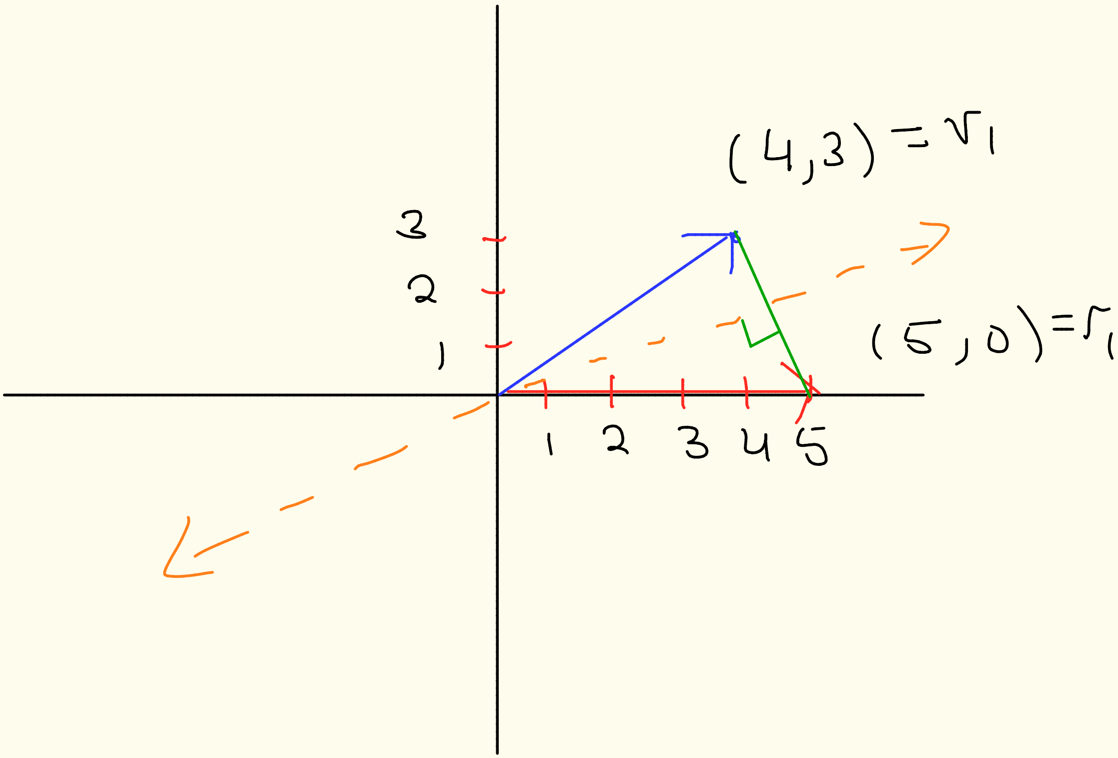
since $5 = \sqrt{4^2+3^2+0^2}$, the two-norm of the original column.

Since the last coordinate of $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ is zero,

we can reduce the dimension by one and send

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ to } \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Picture



orange line = "halfway" between (4,3) and (5,0)

Unitary we're looking for = reflection about orange line.

Intersection of green line w/ orange line is

$$\frac{v_1 + r_1}{2} = x_1$$

$x_1 = P_1 v_1$ where P_1 projects onto the orange line.

So then

$$\frac{v_1 + r_1}{2} = P_1 v_1 .$$

Solving for r_1,

$$r_1 = 2P_1 v_1 - v_1$$

$$= \underbrace{(2P_1 - I_2)}_{\text{unitary } Q_1} v_1$$

$$P_1 = w_1 w_1^*$$

where w_1 is a unit vector in the direction of the orange line.

$$w_1 = \left(\frac{v_1 + r_1}{2}\right) \Big/ \left\| \frac{v_1 + r_1}{2} \right\|_2$$

$$\frac{v_1 + r_1}{2} = \frac{\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}}{2}$$

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$$= \begin{bmatrix} 9/2 \\ 3/2 \end{bmatrix}$$

$$\left\| \frac{v_1 + r_1}{2} \right\|_2 = \frac{1}{2}\sqrt{81 + 9}$$

$$= \frac{1}{2}\sqrt{90} = \frac{3}{2}\sqrt{10}$$

$$w_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

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$$P_1 = w_1 w_1^*$$

$$= \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$$Q = 2P_1 - I_2$$

$$= \frac{1}{5}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} - I_2$$

$$\boxed{= \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}}$$

$$Q_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Abusing notation, write

$$Q_1 = \begin{bmatrix} 4/5 & 3/5 & 0 \\ 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$Q_1 A = \begin{bmatrix} 4/5 & 3/5 & 0 \\ 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 0 \\ 3 & 2 & 1 \\ 0 & 5 & 6 \end{bmatrix}$$

$$Q_1 A = \begin{bmatrix} 4/5 & 3/5 & 0 \\ 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 0 \\ 3 & 2 & 1 \\ 0 & 5 & 6 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 5 & 2/5 & 3/5 \\ 0 & -11/5 & -4/5 \\ 0 & 5 & 6 \end{bmatrix}}$$

Step 2: Send the last two rows of the second column,

$$\begin{bmatrix} -11/5 \\ 5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -11 \\ 25 \end{bmatrix},$$

to

$$\begin{bmatrix} \frac{\sqrt{746}}{5} \\ 0 \end{bmatrix},$$

since $\dfrac{\sqrt{746}}{5} = \left\| \frac{1}{5}\begin{bmatrix} -11 \\ 25 \end{bmatrix} \right\|_2$

Let $v_2 = \frac{1}{5}\begin{bmatrix} -11 \\ 25 \end{bmatrix}$

and $r_2 = \frac{1}{5}\begin{bmatrix} \sqrt{746} \\ 0 \end{bmatrix}$.

Then

$$w_2 = \frac{v_2 + r_2}{2} \Big/ \left\| \frac{v_2 + r_2}{2} \right\|_2$$

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$$\frac{v_2 + r_2}{2} = \frac{1}{10} \begin{bmatrix} -11 + \sqrt{746} \\ 25 \end{bmatrix}$$

$$\left\| \frac{v_2 + r_2}{2} \right\|_2 =$$

$$\frac{1}{10} \sqrt{\underbrace{1492 - 22\sqrt{746}}_{\gamma}}$$

$$w_2 = \frac{1}{\gamma}\begin{bmatrix} -11+\sqrt{746} \\ 25 \end{bmatrix}$$

$$P_2 = w_2 w_2^*$$

$$= \frac{1}{\gamma^2}\begin{bmatrix} -11+\sqrt{746} \\ 25 \end{bmatrix}\begin{bmatrix} -11+\sqrt{746} & 25 \end{bmatrix}$$

$$= \frac{1}{\gamma^2}\begin{bmatrix} 867-22\sqrt{746} & 25(-11+\sqrt{746}) \\ 25(-11+\sqrt{746}) & 625 \end{bmatrix}$$

Then

$$Q_2 = 2P_2 - I_2$$

$$= \frac{2}{\gamma^2}\begin{bmatrix} 867 - 22\sqrt{746} - \frac{\gamma^2}{2} & 25(-11+\sqrt{746}) \\ 25(-11+\sqrt{746}) & 625 - \frac{\gamma^2}{2} \end{bmatrix}$$

Abuse notation again, write

$$Q_2 =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\frac{2}{\gamma^2}\begin{pmatrix} 867-22\sqrt{746}-\frac{\gamma^2}{2} & 25(-11+\sqrt{746}) \\ 25(-11+\sqrt{746}) & 625-\frac{\gamma^2}{2} \end{pmatrix}$}} \\ 0 & & \end{bmatrix}$$

Then apply

$$Q_2 \cdot Q_1 \cdot A$$

hard to compute

$$= \boxed{\begin{bmatrix} 5 & 2/5 & * \\ 0 & \sqrt{746}/5 & * \\ 0 & 0 & * \end{bmatrix}}$$

upper triangular !